Probability II (2.3 - 2.5)

Counting Techniques

1. Equally Likely Outcomes
   (a) For experiments where the outcomes are equally likely; i.e., the same probability is assigned to each simple event, probabilities can be computed by counting.
   (b) If N is the number of outcomes in a sample space and $N(A)$ is the number of outcomes contained in event A, then the probability that event A occurs is $P(A)$, where:
   $$P(A) = \frac{N(A)}{N}$$
   (c) For small N, all outcomes in the sample space, S, can be listed. For large N, not always possible to list all outcomes in S, so we need to make use of **counting rules** such as the product rule, permutations, and combinations.

2. Product Rule
   (a) Product Rule: If the first element, a, of an ordered pair, (a, b), can be selected in n_1 ways, and for each of these n_1 ways the second element, b, can be selected in n_2 ways, then the number of possible ways of selecting ordered pair (a, b), is $n_1 n_2$.
   (b) Example: Consider a two coin toss experiment where there are two possible outcomes, H or T, for each toss.
   - The experimental result will be an ordered pair of outcomes, say $\{T, H\}$, indicating a "tails" on the first toss and a "heads" on the second toss.
   - It can be determined by enumerating all possible outcomes that there are four of them, so that $S = \{HH, HT, TH, TT\}$
   - The number of possible outcomes could also have been computed using the product rule with $n_1 = 2$ and $n_2 = 2$, so that $n_1 n_2 = 4$ possible ordered pairs.

   (c) This product rule is readily extended beyond ordered pairs to ordered sequences of more than two objects.

3. Permutations
   (a) Permutations: Any ordered sequence of k objects taken from a set of n distinct objects is called a **permutation** of size k of the objects.
   (b) Number of Permutations (ordered sequences) of size k taken from n distinct objects:
   $$P_{k,n} = n(n-1)(n-2)\cdots(n-k+1)$$
   - First element can be chosen in n ways
   - Second element can be chosen in $(n-1)$ ways because we're selecting elements without replacement.
   - Third element can be chosen in $(n-2)$ ways, and so on until all k elements are selected.

(c) Factorials: Number of Permutations, $P_{k,n}$, can also be computed from a more condensed, but equivalent, factorial form.

$$P_{k,n} = \frac{n!}{(n-k)!}$$

- For any positive integer, m, $m!$ is read "m factorial", where:

$$m! = m(m-1)(m-2)\cdots(2)1$$

- By definition: $0! = 1$

(d) Example: A boy has four beads. One is red, one is white, one is blue, and one is yellow. How many different ways can three of the beads be strung together in a row?

$$P_{3,4} = \frac{4!}{(4-3)!} = 4! = 24$$

24 different ways

4. Combinations

(a) Combination: Given a set of n distinct objects, any unordered subset of size k of the objects is called a **combination**.

(b) Number of Combinations of size k from n objects can be computed as the Binomial Coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = C_{k,n}$$

(c) For a set of size n, the number of combinations of size k will be smaller than the corresponding number of permutations.

(d) Since for combinations, order is disregarded, there will be several permutations that correspond to a single combination.

(e) Example: A boy has four beads. One is red, one is white, one is blue, and one is yellow. How many different ways can three of the beads be chosen to trade away?

$$C_{3,4} = \binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!} = 4$$

This is a combination since beads are chosen without regard to order.

5. Example: Three balls are selected at random without replacement from a jar containing 3 green balls, 2 red balls, and 3 black balls. What is the probability that of the selected balls one is red and two are black?

$$P(1R\ and\ 2B\ out\ of\ 8) = \frac{n_1 n_2}{N}$$

- n_1 is the number of ways of selecting 1 red ball from 2 red balls
- n_2 is the number of ways of selecting 2 black balls from 3 black balls

- N is the total number of ways of selecting 3 balls from 8 balls

$$n_1 = \binom{2}{1} = \frac{2!}{1!(2-1)!} = 2; \quad n_2 = \binom{3}{2} = \frac{3!}{2!(3-2)!} = 3; \quad N = \binom{8}{3} = \frac{8!}{3!(8-3)!} = 56$$

- The computed probability is then:

$$P(1R \textit{ and } 2B \textit{ out of } 8) = \frac{2(3)}{56} = 0.107$$

6. Example: Suppose the Science Student Council at the University of Southern California consists of one student representative from each of five science departments (Biology, Chemistry, Physics, Mathematics, and Statistics). In how many ways can:

   (a) Both a council president and a vice president be selected?
   - Order important: Selections for different jobs
   - Product rule: $NWAYS = n_1 n_2 = 5(4) = 20$
   - Permutations: $P_{2,5} = \frac{5!}{(5-2)!} = 20$

   (b) A president, a vice president, and a secretary be selected?
   - Order important: Selections for different jobs
   - Product rule: $NWAYS = n_1 n_2 n_3 = 5(4)3 = 60$
   - Permutations: $P_{3,5} = \frac{5!}{(5-3)!} = 60$

   (c) Two members be selected for the Dean's Council?
   - Order not important - both selections for same job
   - Combination: $C_{2,5} = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = 10$

7. Example: Shortly after being put into service, several buses manufactured BusCo Inc. developed cracks on the underside of the main frame. Suppose the city of Moncton has 25 of these buses, and cracks have actually appeared in 8 of them.

   (a) How many ways are there to select a sample of 5 buses from the 25 for a thorough inspection?
   - Order not important

$$C_{5,25} = \binom{25}{5} = \frac{25!}{5!(25-5)!} = 53,130$$

   (b) In how many ways can a sample of 5 buses contain exactly 4 with visible cracks?
   - We want to choose a sample of size 5 containing 4 buses with cracks and 1 without cracks.
   - 25 buses total, 8 with cracks and 17 without cracks.
   - $n_1 \equiv$ (no. ways to pick 4 buses out of 8 with cracks)

$$n_1 = C_{4,8} = \binom{8}{4} = \frac{8!}{4!(8-4)!} = 70$$

- $n_2 \equiv$ (no. ways to pick 1 bus out of 17 without cracks)

$$n_2 = C_{1,17} = \binom{17}{1} = \frac{17!}{1!(17-1)!} = 17$$

- Product rule: $NWAYS = n_1 n_2 = 70(17) = 1,190$

(c) If a sample of 5 buses is chosen at random, what is the probability that exactly 4 of the 5 will have visible cracks?

- P(sample of 5, 4 w/ cracks) $= \frac{n_1}{n_2}$ where:
- $n_1 \equiv$ (no. ways to pick sample of 5 buses, 4 with cracks)

$$n_1 = \binom{8}{4}\binom{17}{1} = 1190$$

- $n_2 \equiv$ (no. ways to pick sample of 5 buses out of 25)

$$n_2 = \binom{25}{5} = \frac{25!}{5!(25-5)!} = 53130$$

- P(sample of 5, 4 w/ cracks) $= \frac{n_1}{n_2} = \frac{1190}{53130} = 0.0224$

(d) If buses are selected as in part (c), what is the probability that at least 4 of those selected will have visible cracks?

- Let X be the number of buses selected containing cracks, then:

$$P(X \geq 4) = P(X = 4) + P(X = 5)$$

where $1 \leq X \leq 5$ for a sample size of 5 buses.

$$P(X \geq 4) = \frac{\binom{8}{4}\binom{17}{1}}{\binom{25}{5}} + \frac{\binom{8}{5}\binom{17}{0}}{\binom{25}{5}} = 0.0234$$

8. Example: Consider the following set containing 3 elements: $\{A, B, C\}$

(a) How many permutations of size 3 can be constructed from the above set?

- Here, $n = 3$ and $k = 3$
- Permutations: $P_{3,3} = \frac{3!}{(3-3)!} = \frac{3 \cdot 2 \cdot 1}{1} = 6$
- So there are 6 ways to order elements A, B, and C

$$(A, B, C),\ (A, C, B),\ (B, A, C),\ (B, C, A),\ (C, A, B),\ (C, B, A)$$

(b) These 6 permutations of $\{A, B, C\}$ are equivalent to a single combination $\{A, B, C\}$

(c) For any combination of size 3, there are 3! permutations obtained by ordering the 3 objects.

$$P_{k,n} = \binom{n}{k} \cdot k!$$

Conditional Probability

1. Introduction

   - Consider some event A, with initially assigned probability, $P(A)$.
   - Then, new information becomes available, i.e., event B occurs.
   - Now we want to know the probability of event A, given that event B has occurred.
   - This is denoted, $P(A|B)$, and is called the Conditional Probability of A, given that event B has occurred.

2. Definition of Conditional Probability

   (a) Conditional Probability: For any two events A and B with $P(B) > 0$, the conditional probability of A given that B has occurred is defined by

   $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

   - This can also be written: $P(A \cap B) = P(B) \cdot P(A|B)$
   - Alternatively, $P(A \cap B) = P(A) \cdot P(B|A)$

   (b) Example: Suppose a country's population consists of three ethnic groups, with each individual belonging to one of four major blood groups. The joint probability table below gives proportions of individuals in the various ethnic group - blood group combinations.

   Blood Group

		O	A	B	AB
	1	.082	.106	.008	.004
Ethnic Group	2	.135	.141	.018	.006
	3	.215	.200	.065	.020

   Compute the following probabilities if an individual is randomly selected from the population. Define events, $A = \{$blood type A selected $\}$, $B = \{$blood type B selected $\}$, and $C = \{$ethnic group 3 selected $\}$.

   i. Calculate $P(A)$, $P(C)$, and $P(A \cap C)$.
      - $P(A) = .106 + .141 + .200 = .447$
      - $P(C) = .215 + .200 + .065 + .020 = .500$
      - $P(A \cap C) = .200$

   ii. Calculate both $P(A|C)$ and $P(C|A)$, and explain in context what each of these probabilities represents.
      - $P(A|C) = \frac{P(A \cap C)}{P(C)} = .200/.500 = .400$

        This is the probability of a person from ethnic group 3 having type A blood.

- $P(C|A) = \frac{P(A \cap C)}{P(A)} = .200/.447 = .447$

  This is the probability of a person with type A blood, being from ethnic group 3.

iii. If the selected individual does not have type B blood, what is the probability that he or she is from ethnic group 1 ?

- Let $D \equiv \{$ethnic group 1 $\}$, and $B^C \equiv \{$not blood type B $\}$
- Since $P(D|B^C) = .082 + .106 + .004 = .192$
- and $P(B^C) = 1 - P(B) = 1 - (.008 + .018 + .065) = .909$

$$P(D|B^C) = \frac{P(D \cap B^C)}{P(B^C)} = .192/.909 = .211$$

(a) Multiplication Rule for $P(A \cap B)$

$$P(A \cap B) = P(B) \cdot P(A|B)$$

- This can also be written: $P(A \cap B) = P(A) \cdot P(B|A)$

(b) Law of Total Probability

- If events A_1, A_2, ... , A_k are both mutually exclusive and exhaustive events
- then, for any other event B,

$$P(B) = \sum_{i=1}^{k} P(B|A_i) \cdot P(A_i)$$

- This can also be written in the form:

$$P(B) = \sum_{i=1}^{k} P(B \cap A_i)$$

(c) Bayes Rule - Allows computation of a posterior probability, $P(A_j|B)$, from given prior, $P(A_i)$, and conditional, $P(B|A_i)$, probabilities.

- If A_1, A_2, ... , A_k is a collection of k mutually exclusive, all $(A_i \cap A_j) = 0$, and exhaustive, $A_1 \cup ... \cup A_k = S$, events with

- $P(A_i) > 0$ for $i = 1, ..., k$

- then, for any other event B for which $P(B) > 0$,

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_i) \cdot P(A_i)}$$

- for $j = 1, ..., k$

(d) Example: A store stocks light bulbs from 3 suppliers. Suppliers A, B, and C supply 10%, 20%, and 30% of the light bulbs respectively. Company A's bulbs have been determined to be 1% defective, while company B's are 3% defective and company C's are 4% defective. If a bulb is selected at random and found to be defective, what is the probability that it came from supplier B?

- let D =defective

$$P(B|D) = \frac{P(B)P(D|B)}{P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)}$$

$$P(B|D) = \frac{0.2(0.03)}{0.1(0.01) + 0.2(0.03) + .07(0.04)} = 0.1714$$

Independence

1. Ordinarily, by the definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

and the conditional probability is some fraction of $P(B)$

(a) Sometimes the occurance of B has no effect on A at all, so that $P(A|B) = P(A)$

(b) **Independent Events:** Events A and B are independent events if $P(A|B) = P(A)$. Otherwise A and B are dependent.

(c) When A and B are independent, the following pairs of events can also be shown to be independent:
A^C and B
A^C and B^C
A and B^C

2. Disjoint Events

(a) When A and B are mutually exclusive (disjoint), $P(A \cap B) = 0$, so that

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = 0$$

- Since $P(A) > 0$,

$$P(A|B) = 0 \neq P(A)$$

- If A and B are mutually exclusive (disjoint), then they cannot be independent.

3. Multiplication Rule for Independent Events

(a) When A and B are independent , so that

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

then

$$P(A \cap B) = P(A) \cdot P(B)$$

(b) For more than two independent events,

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

(c) This relationship can be extended to any number of mutually independent events.

4. Example: Consider rolling 2 fair dice (one red, one green). Let A be the event that the red die shows 3, B the event that the green die shows 4, and C the event that the total of numbers showing on the two dice is 7.

(a) Are events A, B and C pairwise independent (A and B independent, A and C independent, and B and C independent)?

- Let $A \equiv (R = 3)$, $B \equiv (G = 4)$, and $C \equiv (R + G = 7)$
- There are a total of $n = 36$ possible outcomes on the role of 2 dice.
- Event $A \equiv \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$, so that $n(A) = 6$, and

$$P(A) = \frac{n(A)}{n} = \frac{6}{36} = \frac{1}{6}$$

- Event $B \equiv \{(1,4), (2,4), (3,4), (4,4), (5,4), (6,4)\}$, so that $n(B) = 6$, and

$$P(B) = \frac{n(B)}{n} = \frac{6}{36} = \frac{1}{6}$$

- Event $C \equiv \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$, so that $n(C) = 6$, and

$$P(C) = \frac{n(C)}{n} = \frac{6}{36} = \frac{1}{6}$$

- A and B independent?
- $(A \cap B) = \{(3,4)\}$; $P(A \cap B) = \frac{1}{36}$
- For independence: $P(A \cap B) = P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$
- A and B are independent.

- A and C independent?
- $(A \cap C) = \{(3,4)\}$; $P(A \cap C) = \frac{1}{36}$
- For independence: $P(A \cap C) = P(A) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$
- A and C are independent.

- B and C independent?
- $(B \cap C) = \{(3,4)\}$; $P(B \cap C) = \frac{1}{36}$
- For independence: $P(B \cap C) = P(B) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$
- B and C are independent.

(b) Are the three events mutually independent?

- $(A \cap B \cap C) = \{(3,4)\}$; $P(A \cap B \cap C) = \frac{1}{36}$
- For independence: $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$
- $P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$
- so A, B and C are not mutually independent.

5. Example: Overall, 70% of vehicles, examined at the Rotary Esso emissions inspection station, pass the inspection. Assuming that successive vehicles pass or fail independent of one another, calculate the following probabilities:

   (a) $P(\text{all of next 3 vehicles inspected pass}) = \quad ?$
   - $P(P) = 0.70$; $P(F) = 0.30$
   - $P(\text{next 3 pass}) = P(P) \cdot P(P) \cdot P(P) = (0.70)^3 = 0.343$

   (b) $P(\text{at least l of the next 3 vehicles inspected fails}) = \quad ?$
   - $P(\text{at least l of next 3 fails}) = 1 - P(\text{next 3 pass}) = 1 - 0.343 = 0.657$

   (c) $P(\text{exactly 1 of the next 3 vehicles inspected passes}) = \quad ?$
   - $P(\text{exactly 1 of next 3 passes}) = P(PFF) + P(FPF) + P(FFP)$
   - $P(PFF) = P(P) \cdot P(F) \cdot P(F) = 0.7(0.3)(0.3) = 0.063$
   - $P(FPF) = P(F) \cdot P(P) \cdot P(F) = 0.3(0.7)(0.3) = 0.063$
   - $P(FFP) = P(F) \cdot P(F) \cdot P(P) = 0.3(0.3)(0.7) = 0.063$
   - $P(\text{exactly 1 of next 3 passes}) = 3(0.063) = 0.189$

   (d) $P(\text{at most l of the next 3 vehicles inspected passes}) = \quad ?$
   - $P(\text{at most l of next 3 passes}) = P(0\ pass) + P(1\ pass) = P(FFF) + P(\text{exactly 1 of next 3 passes}) = 0.3(0.3)(0.3) + 0.189 = 0.216$

   (e) Given that at least one of the next 3 vehicles passes inspection, what is the probability that all 3 pass? (Note: this is a conditional probability).

$$P(\text{3 pass} \mid \text{1 or more pass}) = \frac{P((3\ pass) \cap (1\ or\ more\ pass))}{P(1\ or\ more\ pass)}$$

   - $(\text{1 or more pass}) \equiv \{1\ pass, 2\ pass, 3\ pass\}$
   - $((3\ pass) \cap (\text{1 or more pass})) = \{3\ pass\}$
   - $(\text{1 or more pass})^C \equiv \{0\ pass\}$
   - $P(\text{1 or more pass}) = 1 - P(0\ pass) = 1 - 0.027 = 0.973$
   - $P((3\ pass) \cap (\text{1 or more pass})) = P(3\ pass) = 0.7(0.7)(0.7) = 0.343$
   - $P(\text{3 pass} \mid \text{1 or more pass}) = \frac{P((3\ pass) \cap (1\ or\ more\ pass))}{P(1\ or\ more\ pass)} = \frac{0.343}{0.973} = 0.353$