Probability I (2.1 - 2.2)

1. Introduction

(a) Probability generally refers to the study of randomness or uncertainty.

(b) In situations where more than one outcome is possible, probability provides a way to quantify the chance, or liklihood, that a particular outcome will occur.

(c) As there are several ways to define probability, we will define **probability** as the proportion of times an outcome would occur in a very long series of repeated independent trials.

(d) Here, we will consider only that portion of probability needed for the study of statistical inference.

2. Sample Space and Events

(a) Sample Space of an Experiment

i. Probability theory is used as a model for situations for which outcomes occur randomly. Such situations are called **experiments**.

ii. The **Sample Space** of an experiment, denoted S, is the set of all possible outcomes of that experiment.

iii. Example: Roll a single die and one of the 6 numbered faces will land face up.

- Experiment: The roll of a single die
- Possible outcomes: 1, 2, 3, 4, 5, 6
- Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$

(b) Events

i. An **event** is a set of outcomes that is a subset of the sample space. Events are usually denoted by capital letters from the beginning of the alphabet.

ii. Event A is said to **occur**, if any of its outcomes occur.

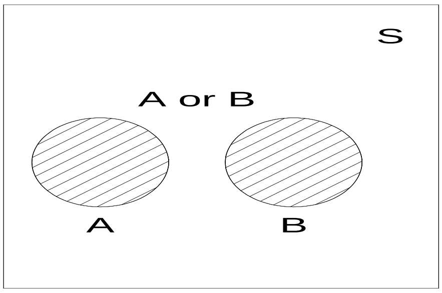
iii. A **simple event,** E_i, consists of only one outcome, whereas a **compound event,** A, consists of more than one outcome.

iv. The probability of compound event A, written $P(A)$, is the sum of the probabilities of the outcomes, E_i, which make up A

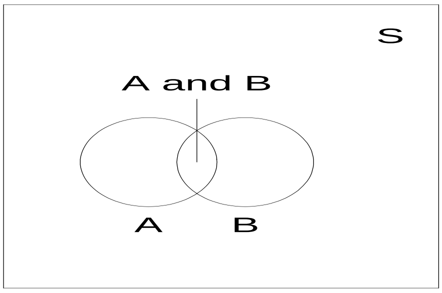
- Experiment: The roll of a single die
- Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$
- Events: $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$

(c) Relations from Set Theory

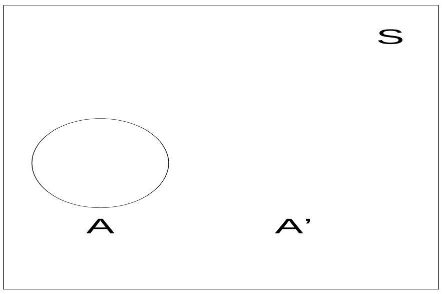
i. **Union:** The union of two events A and B, denoted as $C = A \cup B$, is the event C that either A occurs or B occurs or both occur. The Venn diagram for this union is shown below:



ii. **Intersection:** The intersection of two events A and B, denoted as $C = A \cap B$, is the event C that both A and B occur. This occurs when events A and B share some common elements. The Venn diagram for this intersection is shown below:



iii. **Complement:** The complement of event A, denoted as A^C, is the event that A does not occur. A^C is composed of all elements of the sample space that are not in A. The Venn diagram for complement is shown below:



- Experiment: The roll of a single die
- Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$
- Events: $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$
- Union: $A \bigcup B = \{1, 2, 3, 5\}$
- Intersection: $A \bigcap B = \{1, 3\}$
- Complement: $A^C = \{4, 5, 6\}$

iv. **Disjoint:** Two events A and B are said to be disjoint (or mutually exclusive) if, $(A \cap B)$ is a null set; i.e., there are no outcomes common to both A and B.

- **Experiment:** The roll of a single die
- Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$
- If $A = \{2, 4, 6\}$ (evens) and $B = \{1, 3, 5\}$ (odds)
- Then A and B are mutually exclusive (or disjoint) events.

- **Experiment:** Draw a single card from a standard deck of cards.
- If $A = \{heart, diamond\}$ (red) and $B = \{spade, club\}$ (black)
- Then A and B are mutually exclusive (or disjoint) events.

(d) **Example:** Consider a sample space $S = \{2, 3, 4, 5, 7, 9\}$ on which events $A = \{2, 3, 4, 5\}$, $B = \{3, 5, 7, 9\}$, and $C = \{2, 4, 5, 7, 9\}$ are defined.
Determine the following:

i. $A \cap B = \{3, 5\}$

ii. $A \cup C = \{2, 3, 4, 5, 7, 9\}$

iii. $B \cap C = \{5, 7, 9\}$

iv. $A^C = \{7, 9\}$

v. $B^C = \{2, 4\}$

vi. $A^C \cap B^C = \{\phi\}$

vii. $A^C \cup C = \{7, 9\}$

viii. $A \cup (B \cap C) = \{2, 3, 4, 5, 7, 9\}$

ix. $B \cap (A \cup C) = \{3, 5, 7, 9\}$

(e) **Example:** Consider an experiment where each of 3 cars taking a particular highway exit either turns left (L) or right (R) at the first intersection. In all, there are 8 possible outcomes to this experiment.

Enumeration of Possible Outcomes

Car 1	Car 2	Car 3
L	L	L
L	L	R
L	R	L
L	R	R
R	L	L
R	L	R
R	R	L
R	R	R

i. Sample Space: $S = \{LLL, LLR, LRL, LRR, RLL, RLR, RRL, RRR\}$

ii. Simple Events: $E_1 = \{LLL\}$, $E_2 = \{LLR\}$

iii. Compound Events:
$A = \{RLL, LRL, LLR\}$ = event that exactly one of the cars turns right

$B = \{LLL, RLL, LRL, LLR\}$ = event that at most one of the cars turns right

$C = \{LLL, RRR\}$ = event that all 3 cars turn in the same direction

Note that when LLL occurs:

- simple event E_1 occurs
- compound event B occurs
- compound event C occurs

(f) **Example:** An intersection has two gas stations. Each one has six gas pumps. Consider an experiment in which the number of gas pumps in use at a particular time of day is determined for each station. An experimental outcome specifies the number of pumps in use at the first station (S1) and the number of pumps simultaneously in use at the second station (S2). One possible outcome is $(2, 4)$, another is $(3, 0)$, yet another is $(6, 2)$. In all, there are 49 possible outcomes to this experiment.

Enumeration of Possible Outcomes

		S2						
		0	1	2	3	4	5	6
	0	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
	1	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
S1	3	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,0)	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,0)	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

i. Sample Space: $S = \{(0, 0), (0, 1), ..., (6, 6)\}$

ii. Simple Events: $E_1 = \{(0, 0)\}$, $E_2 = \{(0, 1)\}$, ...

iii. Compound Events: $A = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$ = event that same number of pumps in use at each station

$B = \{(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)\}$ = event that total number of pumps in use at both stations is four

$C = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ = event that at most one pump is in use at each station

3. **Probability Axioms and Properties**

(a) **Axioms of Probability**

i. The axioms of probability are the general assumptions that probability theory is based on.

Axioms of Probability

Axiom 1:	$P(A) \geq 0$	for any event A
Axiom 2:	$P(S) = 1$	for sample space S
Axiom 3:	$P(A_1 \cup A_2 \cup ... \cup A_k) = \sum_{i=1}^{k} P(A_i)$	if all A_i's are disjoint

(b) **Interpreting Probability**

i. The above axioms do not completely determine the assignment of probabilities to events. This depends upon one's interpretation of probability.

ii. Both objective and subjective interpretations are possible

- **Objective** Probabilities: Interpretation rests on a property of the environment.
- **Subjective** Probabilities: Interpretation rests on the opinion of a particular person.

iii. Here, we consider **probability** to be the proportion of times an outcome occurs in a very long series of repeated independent trials.

iv. Consider an experiment repeated many times in an identical and independent fashion.

- Let A be an event consisting of a fixed set of outcomes of the experiment.
- Perform the experiment n times.
- Let $n(A)$ denote the number of times A occurs in the n replicates.
- The relative frequency of occurance of A is then given by $\frac{n(A)}{n}$.
- As n grows large, the relative frequency of occurance of event A stabilizes at a value called the limiting relative frequency of event A.
- The objective interpretation of probability identifies this limiting relative frequency of event A as $P(A)$, the probability of occurance of event A.

$$P(A) = \frac{n(A)}{n}$$

Coin Toss Experiment

Toss (n)	Result (H or T)	Sum H $n(H)$	Rel. Freq. $n(H)/n$
1	H	1	1.00
2	T	1	0.50
3	H	2	0.67
4	H	3	0.75
5	T	3	0.60
6	T	3	0.50
7	H	4	0.57
8	T	4	0.50
9	T	4	0.44
10	H	5	0.50
11	T	5	0.45
12	T	5	0.42
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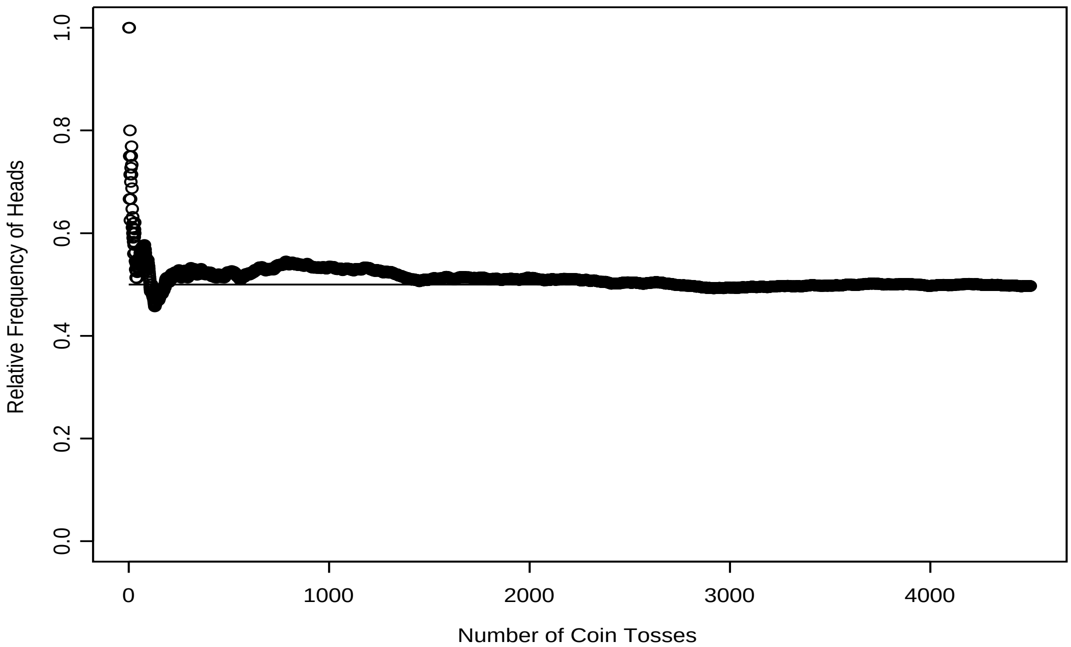


4. **Properties of Probability**

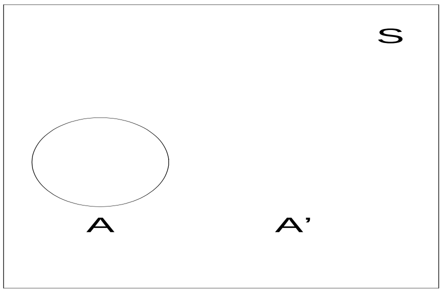
The following relationships can be developed directly from the Axioms.

Properties of Probability

Property 1:	$P(A) = 1 - P(A^C)$	for any event A
Property 2:	$P(A \cap B) = 0$	for disjoint events A and B
Property 3:	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$	for any events A and B

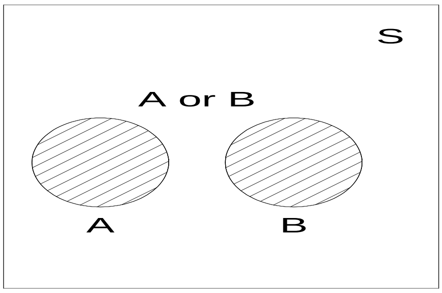
(a) Property 1: $P(A) = 1 - P(A^C)$ for any event A

- By definition of A^C, A and A^C are disjoint and $A \cup A^C = S$
- So that, $P(S) = 1 = P(A \cup A^C) = P(A) + P(A^C)$
- and $P(A) = 1 - P(A^C)$



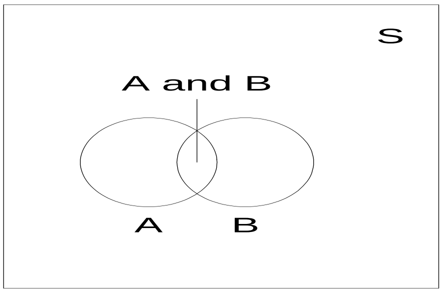
(b) Property 2: $P(A \cap B) = 0$ for disjoint (mutually exclusive) events A and B

- By definition of $A \cap B$ for disjoint (mutually exclusive) events, $A \cap B$ contains no outcomes
- So that $(A \cap B)^C = S$ and $P(S) = P[(A \cap B)^C] = 1 - P(A \cap B)$
- Then, $P(S) = 1 = 1 - P(A \cap B)$ and $P(A \cap B) = 1 - 1 = 0$



(c) Property 3: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any events A and B

- Note that $A \cup B = A \cup (B \cap A^C)$ and that A and $(B \cap A^C)$ are disjoint. This is readily seen from the Venn diagram.
- So that $P(A \cup B) = P(A) + P(B \cap A^C)$
- but note that $B = (B \cap A) \cup (B \cap A^C)$ with $(B \cap A)$ and $(B \cap A^C)$ being mutually exclusive
- So that, $P(B) = P(B \cap A) + P(B \cap A^C)$
- combining, $P(A \cup B) = P(A) + P(B \cap A^C) = P(A) + [P(B) - P(A \cap B)] = P(A) + P(B) - P(A \cap B)$



5. Example: One card is drawn from a well shuffled deck of 52 playing cards. What is the probability that the card drawn is a queen or a hart?

- A 52 card deck contains 4 suits with 13 cards in each suit. There is one queen in each suit. The 4 suits are Hearts, Diamonds, Clubs, and Spades.
- $P(Q \cup H) = ?$
- Probability of drawing a Queen: $P(Q) = \frac{4}{52}$
- Probability of drawing a Heart: $P(H) = \frac{13}{52}$
- Probability of drawing a Queen and a Heart: $P(Q \cap H) = \frac{1}{52}$
- $P(Q \cup H) = P(Q) + P(H) - P(Q \cap H) = \frac{4+13-1}{52} = \frac{16}{52} = 0.308$

6. Example: For a randomly selected Dalhousie student, let A denote the event that he/she has a Visa card and B the event that he/she has a Master card. Suppose we know that for Dalhousie students, $P(A) = 0.5$, $P(B) = 0.4$, $P(A \cap B) = 0.25$.

(a) Compute the probability that the selected student has at least one of the two types of credit cards.

- Having at least one type of credit card, is $(A \cup B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.4 - 0.25 = 0.65$

(b) What is the probability that the selected student has neither type of card?

- Having neither type of credit card, is $(A \cup B)^C$.
- $P[(A \cup B)^C] = 1 - P(A \cup B) = 1 - 0.65 = 0.35$

(c) If the selected student has a Visa card but not a Master card, define this in terms of events A and B. Compute the probability of occurance for this event.

- Having a Visa card but not a Mastercard, is $(A \cap B^C)$.
- $P(A \cap B^C) = P(A) - P(A \cap B) = 0.50 - 0.25 = 0.25$

(d) Example: A box contains four 40-W lightbulbs, five 60-W bulbs, and six 75-W bulbs. If lightbulbs are selected one by one in random order, what is the probability that at least two bulbs must be selected to obtain one that is rated 75-W ?

- Consider the sample space: $S = \{A, B\}$ where A denotes 75-W bulb chosen on first try and B denotes that a 75-W bulb is chosen on 2nd or later try.
- As there are six 75-W bulbs, $P(A) = \frac{6}{15}$
- $P(B) = P(A^C) = 1 - P(A) = 1 - \frac{6}{15} = \frac{9}{15} = 0.60$