Continuous Random Variables II (4.4 - 4.6)

1. **Gamma Distributed Random Variables:** Continuous random variable X is Gamma distributed with parameters α and β if its probability density function (pdf) is:

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-x/\beta} & x \geq 0 \\ 0 & Otherwise \end{cases}$$

where both parameters $\alpha > 0$ and $\beta > 0$.

   (a) **The Gamma Function:** For any $\alpha > 0$, the gamma function is defined as:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} \cdot e^{-x} dx$$

   as a consequence, $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$.

   - When $\alpha = n$, a positive integer, $\Gamma(n) = (n-1)!$. Also, $\Gamma(1/2) = \sqrt{\pi}$.
   - Note that the shape of the gamma distribution changes shape rather dramatically with changes in values of its parameters, α and β.

   (b) **Standard Gamma Random Variables:** The standard gamma random variable has a gamma probability density function with $\beta = 1$. It's pdf is shown below.

$$f(x;\alpha) = \begin{cases} \frac{x^{\alpha-1} \cdot e^{-x}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & Otherwise \end{cases}$$

   Also, when $X > 0$ is a standard gamma rv it's CDF has the form of the **incomplete gamma function** and is given as follows:

$$F(x;\alpha) = \int_0^x \frac{y^{\alpha-1} \cdot e^{-y}}{\Gamma(\alpha)} dy$$

   - $F(x;\alpha)$ values are tabulated in Table A.4 for $1 \leq \alpha \leq 10$ and $1 \leq x \leq 15$. Table values can be used to compute probabilities.
   - β is called a scale parameter as values not equal to 1 either stretch or compress the pdf along the x axis.

   (c) **Cumulative Distribution Function (CDF):** The CDF,$F(x;\alpha,\beta)$,for gamma distributed random variable X, with parameters α and β, is given below for any $x > 0$.

$$P(X \leq x) = F(x;\alpha,\beta) = F\left(\frac{x}{\beta};\alpha\right)$$

   where $F\left(\frac{x}{\beta};\alpha\right)$ is the CDF for the standard gamma random variable noted above.

(d) **Expected Value and Variance** The mean and variance of gamma distributed random variable X, with pdf $f(x;\alpha,\beta)$, are:

$$E(X) = \mu_X = \int_0^\infty x \cdot f(x;\alpha,\beta)dx = \alpha \cdot \beta$$

$$Var(X) = E(X^2) - (E(X))^2 = \sigma_X^2 = \alpha \cdot \beta^2$$

(e) Example: Suppose rv X is gamma distributed with pdf $f(x;\alpha,\beta)$ and parameters $\alpha = 8$ and $\beta = 15$.

a.) Determine $P(60 \leq x \leq 120) = P(x \leq 120) - P(x \leq 60)$.

- $P(X \leq 120) = F(120; 8, 15) = F(\frac{120}{15}; 8) = F(8; 8)$
- $P(X \leq 60) = F(60; 8, 15) = F(\frac{60}{15}; 8) = F(4; 8)$
- $P(60 \leq X \leq 120) = F(8; 8) - F(4; 8) = 0.547 - 0.051 = 0.496$
- Table A.4 values were used above.

b.) Determine the expected value and variance for X.

- $E(X) = \alpha \cdot \beta = 8(15) = 120$
- $Var(X) = \alpha \cdot \beta^2 = 8(15^2) = 1800$

2. **Exponentially Distributed Random Variables:** Continuous random variable X is exponentially distributed with parameter λ if its probability density function (pdf) is:

$$f(x;\lambda) = \begin{cases} \lambda \cdot e^{-\lambda x} & x \geq 0 \\ 0 & Otherwise \end{cases}$$

where parameter $\lambda > 0$.

(a) This **Exponential pdf** is a special case of the gamma pdf with $\alpha = 1$ and $\beta = \frac{1}{\lambda}$.

(b) **Cumulative Distribution Function (CDF):** The CDF, $F(x;\lambda)$, for exponentially distributed random variable X, with parameter λ is given below for any $x \geq 0$.

$$P(X \leq x) = F(x;\lambda) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

(c) **Expected Value and Variance** The mean and variance of exponentially distributed random variable X, with pdf $f(x; \lambda)$, are:

$$E(X) = \int_0^\infty x \cdot f(x; \alpha, \beta) dx = \alpha \cdot \beta = \frac{1}{\lambda}$$

$$Var(X) = \alpha \cdot \beta^2 = \frac{1}{\lambda^2}$$

(d) Example: Let X =the time between two successive arrivals at a drive-up teller window of TD Bank. If X has an exponential distribution with $\lambda = 1$, compute the following:

a.) The expected time between two successive arrivals.

- $E(X) = \alpha \cdot \beta = \frac{1}{\lambda} = 1$

b.) The standard deviation of the time between successive arrivals.

- $Var(X) = \alpha \cdot \beta^2 = \frac{1}{\lambda^2}$
- $SD(X) = \sqrt{Var(X)} = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda} = 1$

c.) $P(X \leq 4)$

- $P(X \leq 4) = 1 - e^{-(1)(4)} = 1 - e^{-4} = 0.982$

d.) $P(2 \leq x \leq 5)$

- $P(2 \leq X \leq 5) = (1 - e^{-1(5)}) - (1 - e^{-1(2)}) = e^{-2} - e^{-5} = 0.129$

3. **Chi-Square Distributed Random Variables:** Continuous random variable X is said to be chi-square distributed with parameter ν if its probability density function (pdf) is the gamma density with $\alpha = \frac{\nu}{2}$ and $\beta = 2$:

$$f(x; \nu) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} \cdot x^{(\nu/2)-1} \cdot e^{-x/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where parameter ν denotes the number of degrees of freedom (df).

(a) As noted above, this **Chi-Square, χ^2, pdf** is a special case of the gamma pdf with $\alpha = \nu/2$ and $\beta = 2$.

(b) It is often used to describe rv X where:

$$X = Y_1^2 + Y_2^2 + ... + Y_n^2 +$$

Here the Y_i are independent rv's, each distributed as $N(0, 1)$.

(c) The chi-Squared random variable and it's distribution are used in a number of procedures in statistical inference that will be considered in more detail later in the course.

4. **Weibull Distributed Random Variables:** Continuous random variable X is weibull distributed with parameters α and β, with $\alpha > 0$ and $\beta > 0$, if its probability density function (pdf) is:

$$f(x;\alpha,\beta) = \begin{cases} \frac{\alpha}{\beta^\alpha} \cdot x^{\alpha-1} \cdot e^{-(x/\beta)^\alpha} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(a) The Weibull pdf is often used as a time-to-failure probability model for electronic components, as is the exponential pdf. The Exponential distribution is a special case of the weibull pdf when $\alpha = 1$ and with exponential parameter $\lambda = \frac{1}{\beta}$.

(b) **Cumulative Distribution Function (CDF):** The CDF, $F(x;\alpha,\beta)$, for Weibull distributed random variable X, with parameters α and β is given below for any $x \geq 0$.

$$P(X \leq x) = F(x;\alpha,\beta) = \begin{cases} 0 & x < 0 \\ 1 - e^{-(x/\beta)^\alpha} & x \geq 0 \end{cases}$$

(c) **Expected Value and Variance** The mean and variance of Weibull distributed random variable X, with pdf $f(x;\alpha,\beta)$, are:

$$E(X) = \mu_X = \beta\Gamma\left(1 + \frac{1}{\alpha}\right)$$

$$Var(X) = \beta^2 \left\{\Gamma\left(1 + \frac{2}{\alpha}\right) - \left[\Gamma\left(1 + \frac{1}{\alpha}\right)\right]^2\right\}$$

(d) Example: If the lifetime X (in hundreds of hours) of a particular type of vacuum tube has a Weibull distribution with parameters $\alpha = 2$ and $\beta = 3$, compute the following:

a.) The expected value, $E(X)$, and variance, $Var(X)$.

- $E(X) = \beta\Gamma(1 + 1/\alpha) = 3\Gamma(1 + 1/2) = 3(\frac{1}{2}\Gamma(1/2)) = \frac{3}{2}\sqrt{\pi} = 2.66$
- $Var(X) = \beta^2 \left\{\Gamma\left(1 + \frac{2}{\alpha}\right) - \left[\Gamma\left(1 + \frac{1}{\alpha}\right)\right]^2\right\} = 9\left[\Gamma(2) - \left(\frac{\sqrt{\pi}}{2}\right)^2\right] = 9(1 - \pi/4) = 1.93$

b.) $P(X \leq 6)$

- $P(X \leq 6) = 1 - e^{-(6/\beta)^\alpha} = 1 - e^{-(6/3)^2} = 1 - e^{-4} = 0.982$

d.) $P(1.5 \leq x \leq 6)$

- $P(1.5 \leq X \leq 6) = P(X \leq 6) - P(X \leq 1.5) = 0.982 - (1 - e^{-(1.5/3)^2}) = 0.760$

5. **Lognormal Distributed Random Variables:** Continuous and nonnegative random variable X is lognormally distributed with parameters μ and σ if $ln(X) \sim N(\mu, \sigma^2)$ its probability density function (pdf) is:

$$f(x; \mu, \sigma) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} \cdot e^{-[ln(x)-\mu]^2/(2\sigma^2)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(a) **Cumulative Distribution Function (CDF):** The CDF, $F(x; \mu, \sigma)$, for lognormally distributed random variable X, with parameters μ and σ is given below for any $x \geq 0$.

$$P(X \leq x) = F(x; \mu, \sigma) = P[ln(X) \leq ln(x)] = P\left(Z \leq \frac{ln(x) - \mu}{\sigma}\right) = \Phi\left(\frac{ln(x) - \mu}{\sigma}\right)$$

(b) **Expected Value and Variance** The mean and variance of lognormally distributed random variable X are:

$$E(X) = e^{\mu + \sigma^2/2}$$

$$Var(X) = e^{2\mu + \sigma^2} \cdot (e^{\sigma^2} - 1)$$

(c) Example: Suppose a microdrill lifetime (number of holes drilled before it breaks), rv X, is lognormally distributed with parameters $\mu = 4.5$ and $\sigma = 0.8$.

a.) What are the mean value and standard deviation of X, microdrill lifetime.

- $E(X) = e^{(\mu + \sigma^2/2)} = e^{(4.5 + (0.8)^2/2)} = e^{4.82} = 123.97$
- $Var(X) = e^{(2\mu + \sigma^2) \cdot (e^{\sigma^2} - 1)} = e^{9.64} \cdot (e^{(0.8)^2} - 1) = e^{4.82} = 13,776.53$
- $SD(X) = \sqrt{Var(X)} = \sqrt{13,776.53} = 117.37$

b.) What is the probability that X is at most 100?

- $P(X \leq 100) = P(Z \leq \frac{ln(100) - 4.5}{0.8}) = P(Z \leq 0.13) = \Phi(0.13) = 0.5517$

c.) What is the probability that X is at least 200? Greater than 200?

- $P(X \geq 200) = P(Z \geq \frac{ln(200) - 4.5}{0.8}) = P(Z \geq 1.00) = 1 - \Phi(1.00) = 0.1587$
- $P(X > 200) = P(X \geq 200) = 0.1587$